

Random discrete Schrödinger operators from Random Matrix Theory

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Abstract. We investigate random, discrete Schrödinger operators which arise naturally in the theory of random matrices, and depend parametrically on Dyson's Coulomb gas inverse temperature β . They are similar to the class of "critical" random Schrödinger operators with random potentials which diminish as $|x|^{-\frac{1}{2}}$. We show that as a function of β they undergo a transition from a regime of (power-law) localized eigenstates with a pure point spectrum for $\beta < 2$ to a regime of extended states with singular continuous spectrum for $\beta \geq 2$.

1. Introduction

Dyson's Coulomb gas model for the spectral fluctuations of random matrix ensembles was recently formulated in terms of ensembles of symmetric, real tridiagonal matrices [1]; see also [2, 3, 4]. These ensembles share the property that the diagonal matrix elements are independent, identically distributed Gaussian random variables, while the off diagonal elements are independent random variables whose probability distribution function (PDF) depends both on the position within the matrix and on the inverse temperature β . We consider these matrix ensembles as ensembles of discrete Schrödinger operators, with random on-site potentials (diagonal matrix elements) and random hopping amplitudes (off-diagonal elements) with prescribed PDF. This viewpoint has been the starting point of recent studies into characterizing the largest eigenvalues of the limiting matrix ensemble in terms of a certain stochastic Schrödinger operator [5, 6].

The interest in these operators stems also from the fact that they are analogous to a class of operators for which the random potential diminishes as a power law $|x|^{-\alpha}$, where x marks the position along the chain. Similar systems were thoroughly discussed in the mathematical literature (see e.g. [7, 8, 9, 10]), where it was proved that a decaying diagonal disorder with $\alpha < 1/2$ induces localization and the spectrum is pure point. However, for $\alpha > 1/2$ the states are extended and the spectrum is absolutely continuous.

The behavior at the critical power $\alpha = 1/2$ depends on the details of the potential, and the eigenstates can be either power-law localized or extended. The model we study here is related to this critical class, but not exactly, since in the present case the transition amplitudes are also random variables. We show that in this model, the parameter β determines the spectral properties: in the regime $0 \leq \beta < 2$ the spectrum is pure - point, and the eigenstates are power law localized, while for $\beta \geq 2$, the eigenstates are extended and the spectrum is singular continuous with a β dependent spectral measure with a Hausdorff dimension $1 - \frac{2}{\beta}$.

We start with a short survey of the relevant information from Random Matrix Theory (RMT). The random matrix ensembles GOE , GUE and GSE are ensembles of $N \times N$ real symmetric, complex hermitian or hermitian real quaternion matrices, respectively, whose matrix elements are independently distributed random Gaussian variables with joint distribution proportional to

$$\exp(-c \operatorname{Tr} H^2). \quad (1)$$

The probability distribution functions of their eigenvalues $\lambda_1, \dots, \lambda_N$ can be written in a concise form

$$P_\beta(\lambda_1, \dots, \lambda_N) = \frac{1}{G_{\beta N}} \exp\left(-\frac{1}{2} \sum_{j=1}^N \lambda_j^2\right) \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k|^\beta. \quad (2)$$

Here, $\beta = 1, 2, 4$ is used for the GOE , GUE , GSE ensembles, respectively, $G_{\beta N}$ are known normalization constants, and c in (1) has been chosen to equal $1/2$, $1/2$, $1/4$ for the GOE , GUE , GSE , respectively. It can be shown from (2), or alternatively directly from the definitions of the ensembles by studying their resolvent, that to leading order the normalized spectral density is supported in the interval $[-\sqrt{2\beta N}, \sqrt{2\beta N}]$ and it assumes the “semi-circle” law

$$\rho(\lambda) = \frac{2}{\pi} \frac{1}{\sqrt{2\beta N}} \sqrt{1 - \frac{\lambda^2}{2\beta N}}. \quad (3)$$

Recently, a systematic way to construct the ensembles corresponding to arbitrary (positive) β was introduced [1], and it is based on the following observation [11, 12, 13]. Any real symmetric matrix $\mathcal{A} \in GOE$ can be orthogonally transformed to a tridiagonal form

$$\mathcal{H}_N = \begin{pmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & a_3 & b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \\ & & & & b_{N-2} & a_{N-1} & b_{N-1} \\ & & & & & b_{N-1} & a_N \end{pmatrix} \quad (4)$$

The probability distribution function of the matrix elements of the corresponding tridiagonal matrix \mathcal{H}_N has the following properties:

- The diagonal elements $\{a_n\}$ are real, independent, identically distributed, Gaussian random variables.
- The off-diagonal elements $\{b_n\}$ are non-negative, independently distributed random variables, with PDF

$$P_{GOE}(b_n) = \chi_n(b_n) \doteq \frac{2}{\Gamma(\frac{n}{2})} (b_n)^{n-1} e^{-b_n^2} . \quad (5)$$

The surprising new result is that by distributing the off diagonal matrix elements using the PDF

$$P_\beta(b_n) = \chi_{\beta n}(b_n) \doteq \frac{2}{\Gamma(\frac{\beta n}{2})} (b_n)^{\beta n-1} e^{-b_n^2} , \quad (6)$$

the eigenvalue PDF of \mathcal{H}_N is given by (2) for any positive β . Thus, the study of the tridiagonal ensembles (denoted by $G\beta E$) provides a convenient way to interpolate between the classical random matrix ensembles with the discrete $\beta = 1, 2, 4$. (A similar method was recently applied in [2] to Dyson's ensembles of unitary matrices, to get Circular β Ensembles; this was used in [14] to calculate eigenvalue statistics for CMV matrices).

Denoting by $\langle \cdot \rangle_\beta$ the expectation value with respect to the $G\beta E$ measures, we can easily find,

$$\langle b_n \rangle_\beta = \frac{\Gamma(\frac{\beta n+1}{2})}{\Gamma(\frac{\beta n}{2})} = \sqrt{\frac{\beta n}{2}} \left(1 - \frac{1}{4\beta n} \right) + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right) \quad (7)$$

and,

$$\langle (b_n - \langle b_n \rangle)^2 \rangle_\beta = \frac{1}{4} + \mathcal{O}\left(\frac{1}{n}\right) . \quad (8)$$

Thus, for large n , the PDF (6) limits to the Dirac distribution $\delta(u_n - 1)$ in the normalized variable defined by $b_n = \sqrt{\frac{\beta n}{2}} u_n$. This also shows that by scaling the matrix elements $\mathcal{H}_N \mapsto \sqrt{2/\beta N} \mathcal{H}_N$, the new off diagonal elements decay as $n^{-1/2}$ where n is counted from the bottom row of the matrix.

Once the matrix \mathcal{A} under consideration is in tridiagonal form (4), a simple recursion relation can be written for the characteristic polynomial $D_N(\lambda) := \det(\lambda I - \mathcal{A}) = \det(\lambda I - \mathcal{H}_N)$. Denoting the determinant of the top $n \times n$ sub-block of $\lambda I - \mathcal{H}_N$ by $D_n(\lambda)$, expansion by the last row shows

$$D_n = (\lambda - a_n) D_{n-1} - b_{n-1}^2 D_{n-2} ; \quad 1 \leq n \leq N , \quad (9)$$

subject to the initial conditions

$$D_{-1} = 0, \quad D_0 = 1 . \quad (10)$$

We remark that by computing the zeros of the characteristic polynomial for the tridiagonal matrices (4) one is sampling from the *correlated* PDF (2).

The matrix \mathcal{H}_N , in the limit $N \rightarrow \infty$ can be considered as a representation of a discrete quantum hamiltonian which governs the dynamics of a quantum particle

hopping randomly between sites on the half line. The distribution of the “on-site potentials” and “hopping amplitudes” are provided by the PDF of the a_n and the b_n respectively. In the mathematics literature, this is referred to as a discrete random Schrödinger operator, or a random Jacobi matrix. We address the following questions: *i.* Whether, for almost all realizations, the eigenfunctions of the random hamiltonian are localized, or in other words, if the spectrum is continuous or discrete. *ii.* In what way the localization depends on the parameter β .

Consider the matrix (4) for a finite N . The eigenvectors $\mathbf{v} = (v_1, \dots, v_N)$ satisfy

$$\mathcal{H}\mathbf{v} = \lambda\mathbf{v} \Rightarrow b_{n-1}v_{n-1} + (a_n - \lambda)v_n + b_nv_{n+1} = 0, \quad \forall \quad 1 \leq n \leq N, \quad (11)$$

with the *boundary* conditions

$$v_0 = v_{N+1} = 0. \quad (12)$$

The homogenous boundary conditions (12) can be satisfied only for N discrete values of λ , and this set coincides with the zeros of the characteristic polynomial $p_N(\lambda)$.

Ignoring the boundary conditions for a while, the recursion relations (11) can be solved for any λ and N , by two independent vectors \mathbf{x} and \mathbf{y} . The Wronskian

$$W(x, y) = b_n(x_{n+1}y_n - x_ny_{n+1}) \quad (13)$$

is independent of n and therefore the growth rate of the vectors compensate each other so that the Wronskian remains constant. It is convenient to chose one of the vectors, say \mathbf{x} as the solution which satisfies the *initial* conditions,

$$x_0 = 0, \quad x_1 = 1. \quad (14)$$

It can be computed (for any λ) by forward iterations of (11). Comparing the two initial value problems (9), (12) and (11), (14), we find that

$$D_n = x_{n+1} \prod_{m=1}^n b_m, \quad (15)$$

which can be proved by direct substitution. Note that the forward iterations usually pick up the solution with the fastest growing rate. An independent solution of the recursion relation can be obtained by imposing the condition $y_{N+1} = 0$, $y_N = 1$ at an arbitrary value of N and perform a backward iteration of (11). This can be done for every λ , and in most cases the solution to be picked up is the one for which $|y_n|$ is the fastest increasing solution when n is decreasing (for $0 \leq n \leq N$). The Wronskian relation (13) implies

$$y_0 = \frac{b_N x_{N+1}}{b_0}. \quad (16)$$

Before addressing the effect of randomness it is useful and instructive to study first the one parametric family of *mean hamiltonians* which are obtained by replacing a_n and b_n in (4) by their $G\beta E$ expectation values. This way we can better appreciate the effect of randomness on the quantum dynamics. We shall show that the eigenfunctions of the mean hamiltonians are extended, and the spectra are absolutely continuous for all $\beta > 0$.

The mean hamiltonians $\langle \mathcal{H} \rangle_\beta$ are tridiagonal matrices with vanishing diagonal matrix elements. The off diagonal terms are given by (7), and, to leading order, are proportional to \sqrt{n} . Thus, for large n , the recursion relations for the components of an eigenvector are:

$$\sqrt{n-1}x_{n-1} + \sqrt{n}x_{n+1} = \sqrt{2} \tilde{\lambda} x_n, \quad (17)$$

where $\tilde{\lambda} = \frac{\lambda}{\sqrt{\beta}}$. The solution of this recursion relation subject to the initial condition $x_0 = 0$, $x_1 = 1$ can be written in terms of the normalized eigenfunctions of the one dimensional harmonic oscillator

$$x_{n+1} = u_n(\tilde{\lambda}) = \left(\frac{1}{\sqrt{\pi n! 2^n}} \right)^{\frac{1}{2}} e^{-\frac{\tilde{\lambda}^2}{2}} H_n(\tilde{\lambda}), \quad (18)$$

with $u_{-1}(\tilde{\lambda}) = 0$. The completeness and orthonormality of the Hermite polynomials implies that for any real λ, μ ,

$$\sum_{m=0}^{\infty} u_m(\lambda) u_m(\mu) = \delta(\lambda - \mu). \quad (19)$$

This proves that the spectrum of the operator $\langle \mathcal{H}_N \rangle_\beta$ for $N \rightarrow \infty$ is absolutely continuous and supported on the entire real line, for all $\beta > 0$. For finite matrices, the boundary condition $v_{N+1} = 0$ is satisfied if $\tilde{\lambda}$ is chosen as one of the zeros of the Hermite polynomial $H_N(\tilde{\lambda})$. For finite but large N the spectrum is located in an interval of size $2\sqrt{2N}$ centered at $\lambda = 0$. The normalized spectral density $\rho(\mu = \tilde{\lambda}/\sqrt{2N})$ is supported on the interval $[-1, 1]$, and approaches the semi-circle law

$$\rho(\mu) = \frac{2}{\pi} \sqrt{1 - \mu^2} \quad (20)$$

in the limit $N \rightarrow \infty$.

In the subsequent paragraphs, we shall show that, in contrast with the eigenfunctions of the mean Schrödinger operators which are delocalized, the eigenfunctions of the disordered operators are power law localized for the $G\beta E$ ensembles with $\beta < 2$. Beyond the critical value $\beta = 2$ the eigenfunctions of \mathcal{H}_N cannot be normalized and the spectrum is continuous. However, the disorder has the effect that now the spectrum is singular continuous with a spectral measure which has a β dependent Hausdorff dimension $1 - \frac{2}{\beta}$.

A prominent quantity of interest in the study of random Schrödinger operators is the mean growth rate of the eigenvectors \mathbf{x} [17]. It is related to the properties of the characteristic polynomial by

$$\mathcal{L}_\beta \doteq \frac{1}{n} \left\langle \log \left| \frac{x_1}{x_{n+1}} \right| \right\rangle_\beta = -\frac{1}{n} \left\langle \log |x_{n+1}| \right\rangle_\beta = -\frac{1}{n} \langle \log |D_n| \rangle_\beta + \frac{1}{n} \sum_{m=1}^n \langle \log |b_m| \rangle_\beta. \quad (21)$$

Thus, the mean Lyapunov exponent \mathcal{L}_β which characterizes the Anderson model, is expressed in terms of the expectation value of the logarithm of the characteristic polynomial of the $G\beta E$ ensemble. Since the latter is known from RMT, and the mean value of the rightmost term in (21) can be evaluated directly, the mean Lyapunov

exponent for this model can be written down for any value of λ . Using the exact PDF for the b_n , we get

$$\frac{1}{n} \sum_{m=1}^n \langle \log |b_m| \rangle = \frac{1}{n} \sum_{m=1}^n \frac{1}{2} \frac{\Gamma'(\frac{m\beta}{2})}{\Gamma(\frac{m\beta}{2})} = \frac{1}{2} \left(\log \frac{n\beta}{2} - 1 + \left(\frac{1}{2} - \frac{1}{\beta} \right) \frac{\log n}{n} \right) + \mathcal{O}\left(\frac{1}{n}\right). \quad (22)$$

This is derived starting with the identity [16]

$$\prod_{j=0}^{N-1} \Gamma(\alpha + 1 + jc) = c^{N(N-1)/2 + N(\alpha+1/2)} (2\pi)^{-N(c-1)/2} \prod_{p=1}^c \prod_{j=0}^{N-1} \Gamma\left(\frac{\alpha + p}{c} + j\right)$$

valid for $c \in \mathbf{Z}_+$. Let

$$f_n(\alpha, c) = c^{\alpha n} \prod_{p=0}^{c-1} \frac{G(n + (\alpha - p)/c + 1) G(-p/c + 1)}{G(n - p/c + 1) G((\alpha - p)/c + 1)},$$

where G denotes the Barnes G -function. Using the asymptotic formula [15]

$$\log \left(\frac{G(N + a + 1)}{G(N + b + 1)} \right) \underset{N \rightarrow \infty}{\sim} (b - a)N + \frac{a - b}{2} \log 2\pi + \left((a - b)N + \frac{a^2 - b^2}{2} \right) \log N + o(1)$$

it follows that

$$f_n(\alpha, c) \sim \exp(\alpha n \log n) c^{\alpha n} e^{-\alpha n} n^{-(c-1)\alpha/2 + \alpha^2/2c} \prod_{p=0}^{c-1} \frac{G(-p/c + 1)}{G((\alpha - p)/c + 1)}.$$

Taking logarithms of both sides, differentiating with respect to α , then setting $\alpha = -1 + c$ gives (22). Furthermore, if c is rational, $c = s/r$ for $s, r \in \mathbf{Z}_+$, the identity [16]

$$f_{rn}(\alpha, s/r) = \prod_{\nu=0}^{r-1} \frac{f_n(\alpha + s\nu/r, s)}{f_n(s\nu/r, s)}$$

proves that (22) remains valid for n a multiple of r . Thus if the limit is to exist for non-integer c it must be given by (22).

The $G\beta E$ expectation value of $\log |D_n(\lambda)|$ is given by

$$\begin{aligned} \frac{1}{n} \langle \log |D_n(\lambda)| \rangle_\beta &= \frac{1}{n} \sum_{l=1}^n \langle \log |\lambda - \lambda_l| \rangle_\beta \\ &= \int dy \rho_\beta(y) \log |y - \lambda| = \frac{1}{2} \left(\log \frac{n\beta}{2} - 1 \right) + \mathcal{O}\left(\frac{1}{n}\right), \end{aligned} \quad (23)$$

where the exact spectral density was replaced by its semi-circle limit (3), and $\lambda \ll \sqrt{2n\beta}$. Substituting in (21), we find that

$$\mathcal{L}_\beta = \frac{\log n^{\frac{1}{2}(\frac{1}{2} - \frac{1}{\beta})}}{n}. \quad (24)$$

Thus, on average, the components of the eigenvectors \mathbf{x} behave as

$$|x_n|^2 \asymp n^{\frac{1}{\beta} - \frac{1}{2}}. \quad (25)$$

Using (16) we expect the other solution of the recursion relation to exhibit a mean decay rate of

$$|y_n|^2 \asymp n^{-\frac{1}{\beta}-\frac{1}{2}}. \quad (26)$$

This result suggests the following scenario: The eigenvectors are square normalizable when $\beta \leq 2$ which would imply that the spectrum is pure point in this β domain. A transition to a continuous spectrum would be expected in the complementary domain. Indeed, with a little more effort, we could show that this is true.

Consider the matrices:

$$S_n^\lambda = \begin{pmatrix} \frac{\lambda - a_n}{b_n} & -\frac{b_{n-1}}{b_n} \\ 1 & 0 \end{pmatrix} \quad (27)$$

and their product

$$T_n^\lambda = S_n^\lambda \cdot S_{n-1}^\lambda \cdots S_1^\lambda. \quad (28)$$

T_n^λ have the property that for the eigenvectors \mathbf{x}

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = T_n^\lambda \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}. \quad (29)$$

In fact, the above holds for any vector satisfying equation (11) (with any boundary conditions). Thus, in order to control the asymptotics of \mathbf{x} , it is reasonable to try and control the asymptotics of $\|T_n^\lambda\|$. By an adaptation of methods from [10] (for details see [18]), it is possible to prove

Proposition 1. For any $\lambda \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \frac{\log \|T_n^\lambda\|^2}{\log n} = \frac{1}{\beta} - \frac{1}{2} \quad (30)$$

with probability one.

With this at hand, a direct application of the methods of [10] (see also [8]) imply

Theorem 1. For any $\lambda \in \mathbb{R}$, with probability one, equation (11) has a solution \mathbf{y} satisfying

$$|y_n|^2 \asymp n^{-(\frac{1}{2}+\frac{1}{\beta})}. \quad (31)$$

Any solution to equation (11) that is linearly independent from \mathbf{y} , satisfies

$$|x_n|^2 \asymp n^{\frac{1}{\beta}-\frac{1}{2}}. \quad (32)$$

The asymptotic behavior of eigenfunctions is intimately connected with the spectral measure of H , associated with the vector $\delta_1 = (1, 0, 0, \dots)$. This is the object defined by

$$d\mu(E) = \text{w-lim}_{N \rightarrow \infty} \sum_{n=1}^N |\langle \delta_1 | \psi_n \rangle|^2 \delta(\lambda - \lambda_n) \quad (33)$$

where λ_n are the eigenvalues of \mathcal{H}_N , and ψ_n are the corresponding normalized eigenvectors. using the technique of spectral averaging (see e.g. [20]), the following can be shown to ensue from the theory of subordinacy ([19]).

Theorem 2. For any β the essential spectrum of H is \mathbb{R} .

If $\beta < 2$, then, with probability one, μ is pure point with eigenfunctions decaying as

$$|x_n|^2 \asymp n^{-(\frac{1}{2} + \frac{1}{\beta})}. \quad (34)$$

If $\beta \geq 2$, then with probability one, for any $\varepsilon > 0$, μ is absolutely continuous with respect to $(1 - \frac{2}{\beta} - \varepsilon)$ -dimensional Hausdorff measure and singular with respect to $(1 - \frac{2}{\beta} + \varepsilon)$ -dimensional Hausdorff measure.

Thus, we see that, as long as $\beta < 2$, H has square-summable eigenfunctions whereas for $\beta \geq 2$ the spectrum is purely continuous. This spectral transition at $\beta = 2$ from pure-point to continuous spectrum is, in a certain sense, continuous in β , since the decay rate of the eigenvalues $(\frac{1}{2} + \frac{1}{\beta})$ changes continuously in β . This is in striking contrast with the transition expected in the Anderson model. Thus, for the case studied here, the Inverse Participation Ratio, for example, should change continuously from 1 (when $\beta = 0$) to 0 (for $\beta = \infty$).

An interesting feature of this continuous transition is the connection between the ‘extendedness’ of states and the level repulsion observed on finite scales. Equation (2) above implies that the probability of finding pairs of close eigenvalues, of the finite dimensional matrix \mathcal{H}_N , diminishes as β increases. Theorem 1 says that as β increases, the states decay at a slower rate. These two facts are complementary: We expect slower decay rate to be connected with stronger level repulsion. Our analysis confirms this expectation and gives physical meaning to this aspect of the eigenvalue statistics of $G\beta E$.

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